

Classical Symmetric Top in a Gravitational Field

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Index Terms

symmetric top, gyroscope, precession, nutation, Lagrangian formulation, mathematical physics

Abstract

The analysis of the symmetrical top is a fascinating topic in classical mechanics. It is a simple system that exhibits counter-intuitive behaviour. It's famous "gravity-defying" behaviour is well known. This brief paper first explores the Lagrangian formulation of the symmetrical top in a uniform force-field (e.g. gravity). After establishing a few constraints, a dynamic model is presented. From this model, precession rates and nutation behaviour are deduced.

I. INTRODUCTION: THE LAGRANGIAN FORMULATION

Most people are familiar with the bizarre balancing behaviour of the spinning top. Why causes the heavy spinning wheel to remain perched on its pointed stem, in apparent defiance of the pull of gravity? How can we develop a mathematical model of this behaviour? This paper will detail a methodology for quantifying the mechanics of the spinning symmetric top (gyroscope in a gravitational field).

The geometry of the symmetric gyroscopic top in a gravitational field consists of a "heavy" wheel on a narrow stem whose bottom point is fixed to the origin, but is free to rotate (as shown in Figure 1).

The z' axis passes through the center of rotation of the top. The motion of the top can be expressed in terms of the so-called Eulerian angles (ϕ, θ, ψ) . The variable θ is the angle that the z' axis makes with the stationary z axis. The variable ϕ is the angle that is formed by the projection of the z' axis in the $x - y$ plane and the x axis. Counter-clockwise rotation yields positive angles. The variable ψ is the angular position around the z' axis. Since our top is circularly symmetric, the choice of origin for ψ is arbitrary. We will only be concerned with the rate of angular rotation of the wheel about z' , given by $\dot{\psi}$. Other dynamical variables of interest are the time derivatives of ϕ and θ . These variables $\dot{\phi}$ and $\dot{\theta}$ describe the precession and nutation of the top. The length of the stem is l . We assume the mass of the stem is negligible with respect to that of the wheel. The acceleration due to gravity is g and is assumed to point "downwards" along the $-z$ axis.

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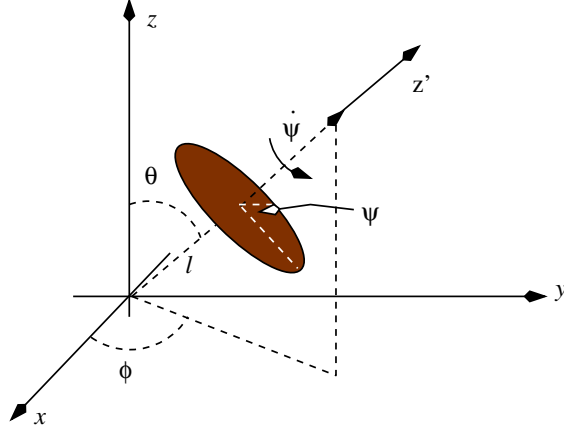


Fig. 1. Illustration of the symmetric top on a long stem. Note the orientation of the axes.

The most convenient way to derive the equations of motion for this system is to use the Lagrangian or Hamiltonian formalism. We will focus on the Lagrangian method in this paper. This formulation is readily converted to the Hamiltonian form using a simple transformation that is left to the reader to explore.

The Lagrangian of the top is deceptively simple, as seen in (1):

$$L = \frac{1}{2}I_{\psi}(\dot{\psi} + \dot{\phi} \cos \theta)^2 + \frac{1}{2}I_0(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) - mgl \cos \theta \quad (1)$$

The first term is the contribution of the rapidly spinning wheel to the kinetic energy of the top. I_{ψ} is the moment of inertia for rotation around the z' axis ($= ma^2/2$ for a disc of radius a and mass m , ignoring the mass of the stem). The second term is the part of the kinetic energy yielded by rotational motion around the pivot point at the origin. I_0 is the moment of inertia of the whole structure about the x or y axis. For the wheel on a stem of length l , we have

$$I_0 = ml^2 + ma^2/4 \quad (2)$$

The last term in (1) refers to the gravitational potential energy of the whole system, where the system center of mass is a height $l \cos \theta$ above the pivot point. We see that the Lagrangian does not depend on the variables ψ , ϕ or t . As a consequence, the angular momenta p_{ϕ} and p_{ψ} and the total energy E are invariant (i.e. conserved).

Utilizing the relationship

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_i} \right) - \frac{\partial L}{\partial x_i} = f_i \quad (3)$$

where x_i represents the system canonical coordinates ϕ , ψ and θ ; f_i are externally applied forces (torques), here assumed to vanish. Applying (3) to (1), we deduce the following conservation laws:

$$\frac{\partial L}{\partial \dot{\psi}} = I_{\psi}(\dot{\psi} + \dot{\phi} \cos \theta) = p_{\psi} \quad (4)$$

$$\frac{\partial L}{\partial \dot{\phi}} = I_{\psi}(\dot{\psi} + \dot{\phi} \cos \theta) \cos \theta + I_0 \dot{\phi} \sin^2 \theta = p_{\phi} \quad (5)$$

$$E = \frac{1}{2}I_\psi (\dot{\psi} + \dot{\phi} \cos \theta)^2 + \frac{1}{2}I_0 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + mgl \cos \theta \quad (6)$$

The values p_ψ , p_ϕ represent the angular momenta about the ψ and ϕ axes as well as the system total energy E . These are constants of motion and, as seen below, are indispensable quantities for deriving other aspects of the top's motion such as precession and nutation. The complete set of system state variables are completely integrable.

II. PRECESSION

The reader is probably familiar with the gyroscope's slow circling behaviour as it apparently defies gravity in its odd leaning without falling over. This is precession: movement through the azimuthal angle ϕ . For the moment, we will ignore nutation $\dot{\theta} = 0$ and focus on the leaning angle θ , assumed constant, and the rate of precession $\dot{\phi}$.

We shall use the Energy Method to derive the precession rate. This entails finding a point where the total system energy is stationary with respect to the angle θ , i.e.

$$\frac{\partial E}{\partial \theta} = 0 \quad (7)$$

We can do this because E is a conserved quantity set by the initial conditions. The energy must be a "stationary" or extreme value equal to the total system energy. Any deviation in θ from the correct value represents an error in the solution that pushes away from stationary value. This provides us with a handy mathematical trick for finding important physical properties of the system.

Since we ignore nutation (only steady precession is assumed), we eliminate the variables $\dot{\psi}$ and $\dot{\phi}$ in E by putting E in terms of the invariants using (4) and (5). We get:

$$E = \frac{p_\psi^2}{2I_\psi} + \frac{(p_\phi - p_\psi \cos \theta)^2}{2I_0 \sin^2 \theta} + mgl \cos \theta. \quad (8)$$

Taking the derivative of E with respect to the elevation angle θ yields

$$\frac{\partial E}{\partial \theta} = \frac{(p_\phi - p_\psi \cos \theta) p_\psi \sin^3 \theta - \sin \theta \cos \theta (p_\phi - p_\psi \cos \theta)^2}{I_0 \sin^4 \theta} - mgl \sin \theta \quad (9)$$

Solution of (9) for θ yields the angle at which the gyroscope will lean given the speed of rotation of the wheel. The angle θ is found by solving (9) for the roots in terms of the invariant quantities.

The precession rate is found by reorganising (5) into

$$\dot{\phi} = \frac{p_\phi - p_\psi \cos \theta}{I_0 \sin^2 \theta}. \quad (10)$$

The relationship in (9) can be rearranged by grouping terms in $p_\phi - p_\psi \cos \theta$ to read

$$\frac{\cos \theta}{p_\psi \sin^2 \theta} (p_\phi - p_\psi \cos \theta)^2 - (p_\phi - p_\psi \cos \theta) + \frac{mgl I_0 \sin^2 \theta}{p_\psi} = 0. \quad (11)$$

Using the quadratic formula to solve for the roots of this equation, we have

$$p_\phi - p_\psi \cos \theta = \frac{p_\psi \sin^2 \theta}{2 \cos \theta} \left(1 \pm \sqrt{1 - \frac{4mgl I_0 \cos \theta}{p_\psi^2}} \right). \quad (12)$$

If we use (10) in (12), an explicit expression for the precession rate is produced in terms of the invariants and the angle θ :

$$\dot{\phi} = \frac{p_\psi}{2I_0 \cos \theta} \left(1 \pm \sqrt{1 - \frac{4mglI_0 \cos \theta}{p_\psi^2}} \right). \quad (13)$$

Note that if $|p_\psi| < \sqrt{4mglI_0 \cos \theta}$, there is no well defined uniform precession. The top starts to behave more like a swinging pendulum than the familiar gyroscope. In fact, in the limit of no wheel spinning, the equations of motion become exactly those of the suspended pendulum.

The quadratic equation produces two solutions: the so-called ‘‘fast’’ precession and the ‘‘slow’’ precession solutions. If the rotational speed of the heavy wheel is large (i.e. $p_\psi \gg \sqrt{4mgl \cos \theta}$), the fast precession rate is

$$\dot{\phi}_{\text{fast}} \approx \frac{p_\psi}{I_0 \cos \theta} = \frac{I_\psi \omega}{I_0 \cos \theta}. \quad (14)$$

Incidentally, the rotational speed of the wheel (usually much higher than the precession speed) is given by

$$\omega = \frac{p_\psi}{I_\psi} = \dot{\psi} - \dot{\phi} \cos \theta. \quad (15)$$

This expression is simply a recasting of (4). Note that (15) contains contributions from both the high speed wheel rotation about the z' axis ($\dot{\psi}$) as well as the precession motion around the z axis ($\dot{\phi}$). This is because the z and z' axes are generally not perpendicular to each other.

The slow precession rate is found by using the small argument approximation for $\sqrt{1-x}$:

$$\sqrt{1-x} \approx 1 - \frac{1}{2}x \quad (16)$$

Hence, we find that

$$\dot{\phi}_{\text{slow}} \approx \frac{mgl}{p_\psi} = \frac{mgl}{I_\psi \omega} \quad (17)$$

It is interesting that the slow precession rate does not depend on the ‘‘leaning angle’’ θ . The gyroscope will precess at a rate solely based on the wheel speed, stem length and the the gravitational pull on the center of mass of the gyroscope. The slow precession rate is the one most commonly observed experimentally.

Now that we have an expression for the precession rate, by using (10), we can find the ‘‘lean angle’’ in terms of the system invariants. Solving for $\cos \theta$ in terms of $\dot{\phi}$, we get:

$$0 = (I_0 \dot{\phi} - p_\phi) + p_\psi \cos \theta - I_0 \dot{\phi} \cos^2 \theta \quad (18)$$

Using the ‘‘slow’’ precession rate, i.e. $\dot{\phi} = mgl/p_\psi$ in (18) and the quadratic equation to solve for $\cos \theta$, we see that

$$\cos \theta = \frac{p_\psi^2}{2I_0 mgl} \left(1 \pm \sqrt{1 + \left(\frac{4I_0 mgl}{p_\psi^2} \right) \left(\frac{I_0 mgl/p_\psi - p_\phi}{p_\psi} \right)} \right) \quad (19)$$

Only the solution with the minus sign gives us a meaningful solution, because $\cos \theta$ can only take values between -1 and 1. Moreover, if $p_\psi^2 \gg I_0 mgl$, we can use the small argument approximation for the square-root, which gives us:

$$\cos \theta \approx -\frac{I_0 mgl}{p_\psi^2} + \frac{p_\phi}{p_\psi}. \quad (20)$$

Again, since p_ψ is very large, this reduces further to

$$\cos \theta_{\text{lean}} \approx \frac{p_\phi}{p_\psi} \quad (21)$$

III. NUTATION

If we rewrite (8) including the kinetic energy of θ -directed motion (nutation), we have:

$$E = \frac{p_\psi^2}{2I_\psi} + \frac{(p_\phi - p_\psi \cos \theta)^2}{2I_0 \sin^2 \theta} + \frac{1}{2}I_0 \dot{\theta}^2 + mgl \cos \theta. \quad (22)$$

A “potential energy” function that provides a “restoring force” can be written as

$$V(\theta) = \frac{p_\psi^2}{2I_\psi} + \frac{(p_\phi - p_\psi \cos \theta)^2}{2I_0 \sin^2 \theta} + mgl \cos \theta. \quad (23)$$

The magnitude of this velocity is found from the kinetic energy to be:

$$|\dot{\theta}| = \sqrt{\frac{2}{I_0} (E - V(\theta))}. \quad (24)$$

The extreme values of θ can be found by solving for the roots of $E - V(\theta)$ in θ (these are the points in θ where $\dot{\theta}$ vanishes; i.e. the nutation angle extremes):

$$E - \frac{(p_\phi - p_\psi \cos \theta)^2}{2I_0 \sin^2 \theta} - \frac{p_\psi^2}{2I_\psi} - mgl \cos \theta = 0. \quad (25)$$

Setting $x = \cos \theta$ and reorganising into a cubic polynomial equation, we have:

$$\left\{ 2I_0 \left(E - \frac{p_\psi^2}{2I_\psi} \right) - p_\phi^2 \right\} + (2p_\phi p_\psi - 2I_0 mgl)x - \left\{ 2I_0 \left(E - \frac{p_\psi^2}{2I_\psi} \right) + p_\phi^2 \right\} x^2 + 2I_0 mgl x^3 = 0 \quad (26)$$

By solving for x in (26), we find one or two roots between -1 and 1 and another non-physical root ($|x| > 1$). The two physically meaningful roots will yield the span of angles of the nutation. If the roots are equivalent, there is no nutation; only smooth precession.

IV. EQUATIONS OF MOTION

The equations of motion are summarised below. Equation (29) can be solved using a numerical method, using (31). The remaining equations are immediately integrated based on the solution for $\theta(t)$ and the initial system state. Unlike in the previous sections, there are no approximations. By constructing a numerical solver, we can see if the approximations we made in the previous sections are valid.

$$\dot{\phi} = \frac{p_\phi - p_\psi \cos \theta}{I_0 \sin^2 \theta} \quad (27)$$

$$\dot{\psi} = \frac{p_\psi}{I_\psi} + \frac{p_\phi - p_\psi \cos \theta}{I_0 \sin^2 \theta} \cos \theta \quad (28)$$

$$I_0 \dot{\eta} = -\frac{\partial V}{\partial \theta} \quad (29)$$

$$\dot{\theta} = \eta \quad (30)$$

$$\frac{\partial V}{\partial \theta} = \frac{(p_\phi - p_\psi \cos \theta) p_\psi}{I_0 \sin \theta} - \frac{\cos \theta (p_\phi - p_\psi \cos \theta)^2}{I_0 \sin^3 \theta} - mgl \sin \theta \quad (31)$$

This system of first order, nonlinear ordinary differential equations can be integrated in time using the Octave [1] ODE solver LSODE using the code below.

```

clear all;                                # Start with a clean slate
global p_psi p_phi I0 mgl; # Make these variables visible to function
phi_dot = 0.0                             # Initial condition on azimuthal velocity
psi_dot = 200.0 * pi                       # Initial wheel spinning speed
theta0 = pi/3.0                             # Initial wheel elevation angle
I0 = 2.33e-3                                # Moment of inertia around bottom pivot point
I_psi = 1.25e-4                             # Moment of inertia for heavy wheel around Z' axis
m = 0.1                                     # Mass of top
g = 9.81                                    # Acceleration due to gravity
l = 0.15                                    # Length of stem
mgl = m * g * l;                            # product of m, g and l
p_psi = I_psi * (psi_dot + phi_dot * cos(theta0)) #Compute and print p_psi
p_phi = p_psi * cos(theta0) + I0 * phi_dot * sin(theta0)^2 #Compute and print p_phi
precess = mgl / p_psi                        #Compute slow precession speed value
#Complete system energy
E = 0.5 * p_psi * p_psi / I_psi +
    0.5 * I0 * phi_dot ^2 * sin(theta0)^2 + mgl * cos(theta0)
# compute polynomial coefficients to solve for
# roots to cubic eqn for nutation excursion
c0 = 2.0 * I0 * (E - p_psi^2 / (2.0 * I_psi)) - p_phi^2;
c1 = 2.0 * p_phi * p_psi - 2.0 * I0 * mgl;
c2 = -2.0 * I0 * (E - p_psi^2 / (2.0 * I_psi)) - p_psi^2;
c3 = 2.0 * I0 * mgl;
# points for plotting restoring force function
a = linspace(0,pi,251);
# Restoring force function for plotting
ke = c0 + c1 * cos(a) + c2 * cos(a).^2 + c3 * cos(a).^3;
figure(1)
plot(a,ke,'') # plot!
grid

# Find roots that give nutation extremes
r = roots([c3, c2, c1, c0])
for i = 1:3
    if(abs(r(i)) <= 1.0)
        acos(r(i)) # Compute angles
    endif
endfor
# Generate time sequence for solving ODE
t = linspace(0, 3, 1001);
# Solve ODE for full theta/phi motion
[x, istate, msg] = lsode("xdot", [0.; theta0; 0.0], t);

```

```

figure(2)
plot(x(1:1001,3), x(1:1001,2),''); # Plot theta/phi trajectory
grid
figure(3);
plot(t, x(1:1001,3),'');          # Plot precession path
grid

```

We also need an Octave function that defines the system of differential equations “ \dot{x} ”:

```

#      | omega |
#      |       |
#      | theta |
#x =   |       |
#      | phi   |

function y = xdot(x,t)
global p_phi p_psi I0 mgl; #use these global values
y = [0.; 0.; 0.];
y(1) += (p_phi * cos(x(2)) - p_psi) *
        (p_psi * cos(x(2)) - p_phi) / (I0 * sin(x(2))^3);
y(1) -= mgl * sin(x(2));
y(2) = -x(1) / I0;
y(3) = (p_phi - p_psi * cos(x(2))) / (I0 * sin(x(2))^2);
endfunction

```

We are now ready to compute some examples.

V. EXAMPLE NUMERICAL SIMULATIONS

Let us consider a simple example to get a feel for the order of magnitude of the motion experienced by the spinning top. Table I gives the initial state of the top as well as the various fixed parameters.

TABLE I

TABLE OF PARAMETERS FOR CUSP NUTATION

mom. of inertia 1	I_ψ	$1.25 \times 10^{-1} \text{ kg}\cdot\text{m}^2$	mom. of inertia 2	I_0	$2.33 \times 10^{-3} \text{ kg}\cdot\text{m}^2$
wheel mass	m	0.1kg	wheel ang. mom.	p_ψ	$0.0392 \text{ kg}\cdot\text{m}^2\text{s}^{-1}$
wheel radius	a	0.05m	azimuthal ang. mom.	p_ϕ	$0.0273 \text{ kg}\cdot\text{m}^2\text{s}^{-1}$
stem length	l	0.15m	Total energy	E	6.27J
wheel spin rate	$\dot{\psi}_0$	100 π rad/s	Lower nut. ang.	θ_l	0.785 ($\pi/4$) rad
initial azimuthal rate	$\dot{\phi}_0$	0 rad/s	Upper nut. ang.	θ_h	1.26 rad
initial lean angle	θ_0	$\pi/4$ rad	precession rate	$\dot{\phi}$	3.74rad/s

If we wish to know the steady precession rate (only attainable if the initial conditions are correct such that no

nutaton takes place, or the nutaton decays by slight frictional losses), we use (17) to find

$$\dot{\phi} = \frac{(0.1kg)(9.81 m/s^2)(0.15m)}{(0.0392 kgm^2/s)} = 3.7 s^{-1}, \quad (32)$$

or just over a half a revolution every second.

It should be noted that this precession rate represents the average value for swept-out azimuthal angle even in the presence of nutation [1].

In order to find the range of angles θ visited by the motion of nutation, we find the roots of (26). Given our set of conditions we find two physical roots that give us angles of 0.785 and 1.26 radians. This indicates that the top oscillates between the original (starting) lean angle of $\pi/4 \approx 0.78$ and 1.3; that is, over a span of about 0.55 radians. This is exactly what is seen in Figure 2. In practice, these wobbles tend to die out as a result of slight frictional losses leaving only the rapidly spinning wheel and the precessional movement. In these simulations, we have included no frictional loss.

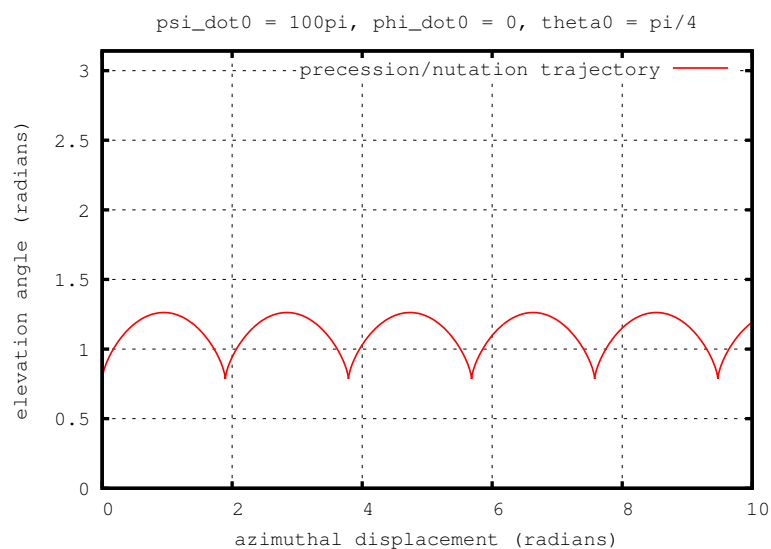


Fig. 2. Nutation/precession trajectory of top with parameters in Table I.

Since we chose the initial conditions such that the azimuthal velocity vanishes at the minimum nutation angle, we get a cusp in the trajectory at $\theta = \pi/3$ (or $\theta = 1.047$). This cusp vanishes if we have a slight “forward” movement ($\dot{\phi} \neq 0$) as laid out in Table (II).

Here, we start with a 3 rad/s azimuthal angular velocity in the direction of precession. The plot in Figure 3 no longer exhibits the cusps, but a smooth trajectory that looks almost sinusoidal.

In the the third example, we start off with a little bit of retrograde azimuthal motion. We set the initial value $\dot{\phi}_0 = -3$ rad/s.

The nutation/precession trajectory now follows a “loopy” path as the top azimuthal velocity periodically goes negative.

TABLE II

TABLE OF PARAMETERS FOR SMOOTH NUTATION

mom. of inertia 1	I_ψ	$1.25 \times 10^{-1} \text{ kg}\cdot\text{m}^2$	mom. of inertia 2	I_0	$2.33 \times 10^{-3} \text{ kg}\cdot\text{m}^2$
wheel mass	m	0.1kg	wheel ang. mom.	p_ψ	$0.0395 \text{ kg}\cdot\text{m}^2\text{s}^{-1}$
wheel radius	a	0.05m	azimuthal ang. mom.	p_ϕ	$0.0314 \text{ kg}\cdot\text{m}^2\text{s}^{-1}$
stem length	l	0.15m	Total energy	E	6.36J
wheel spin rate	$\dot{\psi}_0$	$100\pi\text{rad/s}$	Lower nut. ang.	θ_l	$0.785 (\pi/4)$
initial azimuthal rate	$\dot{\phi}_0$	3 rad/s	Upper nut. ang.	θ_h	0.969
initial lean angle	θ_0	$\pi/4$ rad			

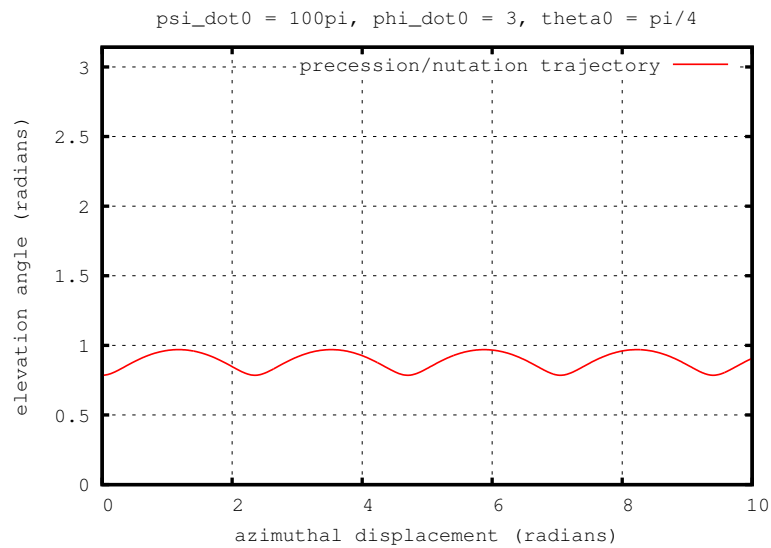


Fig. 3. Nutation/precession trajectory of top with parameters in Table II.

TABLE III

TABLE OF PARAMETERS FOR SMOOTH RETROGRADE NUTATION

mom. of inertia 1	I_ψ	$1.25 \times 10^{-1} \text{ kg}\cdot\text{m}^2$	mom. of inertia 2	I_0	$2.33 \times 10^{-3} \text{ kg}\cdot\text{m}^2$
wheel mass	m	0.1kg	wheel ang. mom.	p_ψ	$0.0390 \text{ kg}\cdot\text{m}^2\text{s}^{-1}$
wheel radius	a	0.05m	azimuthal ang. mom.	p_ϕ	$0.0241 \text{ kg}\cdot\text{m}^2\text{s}^{-1}$
stem length	l	0.15m	Total energy	E	6.19J
wheel spin rate	$\dot{\psi}_0$	$100\pi\text{rad/s}$	Lower nut. ang.	θ_l	$0.785 (\pi/4)$
initial azimuthal rate	$\dot{\phi}_0$	-3 rad/s	Upper nut. ang.	θ_h	1.504
initial lean angle	θ_0	$\pi/4$ rad			

VI. LAST WORDS

In this brief paper we have attempted to summarise the physics of the rapidly spinning symmetric top using a classical Lagrangian approach. By identifying a set of invariants, it is possible to deduce the mechanical behaviour

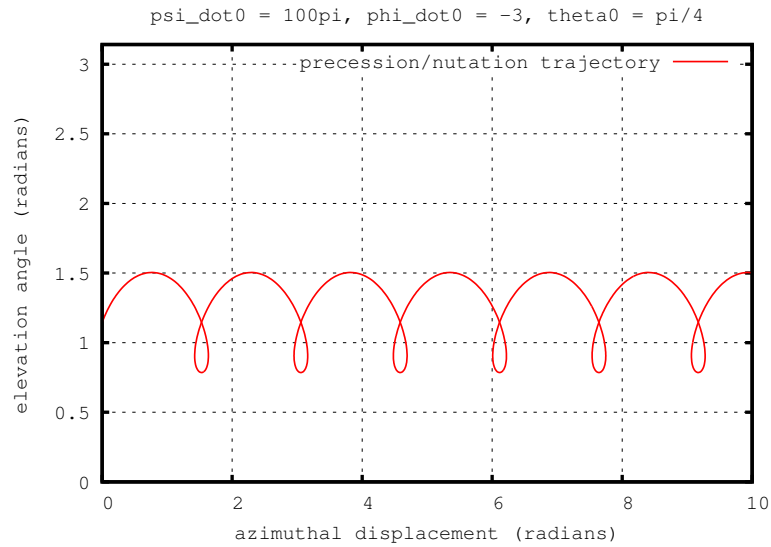


Fig. 4. Nutation/precession trajectory of top with parameters in Table III.

of the top's gyroscopic action without having to resort to full numerical (or analytic) solution of the equations of motion. A numerical model (based on the Gnu Octave LSODE solver) for the nutation and precession is included to verify the calculated behavior.

REFERENCES

- [1] Gnu Octave, <http://www.gnu.org/software/octave/>
- [2] K. Symon, *Mechanics*, Third edition, Addison Wesley, 1972.
- [3] L. Landau and E. Lifshitz, *Course in Theoretical Physics: Mechanics*, Vol. I, Elsevier, 2008 reprint.
- [4] C. Lanczos, *The Variational Principles of Mechanics*, Fourth edition, Dover, 1970.

Corrections and modifications

20120615: Added missing factors of 1/2 in Lagrangian. Replaced I_ψ with I_0 in θ equation of motion

20120707: Cleaned up text, equations; added numerical model.